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# CAUSAL BEHAVIOUR OF FIELD THEORIES WITH NON-LOCALIZABLE INTERACTIONS

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The causal behaviour of field theories with non-localizable interactions of the Kristensen-Møller type is discussed in the perturbation approximation, with particular attention to interactions involving only particles with time-like momentum vectors. Causal behaviour is understood to imply that all observable particles of positive energy are propagated at a velocity less than the velocity of light. It is shown that the causal behaviour of the non-local interaction theories is determined both by the location of the singularities of the propagation function, and by the continuity of the various derivatives of the form function. It is further demonstrated that, by choosing these derivatives to be continuous in sufficiently high orders, the probability of observing signals propagating with a velocity greater than that of light may be made to decrease more rapidly than any arbitrary inverse power of the distance between the points at which the signal is observed. The relation of this work to other treatments of causality is discussed.

## 1. Introduction.

Considerable interest has recently been attached to discussions of field theories involving non-local interaction, that is, field theories in which the interaction term in the Lagrangian involves the field variables at different points in space and time. Following Kristensen and Møller(1), this interaction term may be written as[1]

$$L_{\text{int}} = -\int d^4x'\, d^4x''\, d^4x'''\, \psi^+(x')\, \Phi(x'x''x''')\, \varphi(x'')\, \psi(x'''), \quad (1.1)$$

in which $\Phi(x'x''x''')$ is the form factor of the interaction and will in general be a product of a matrix operator and a function

[1] We shall use the notation $F(123)$ for $F(x^\mu_1, x^\mu_2, x^\mu_3)$. The adjoint field is denoted by $\psi^+$, and may be taken as $\psi^+ = \psi^* \gamma_4$, with $\psi^*$ the Hermitian conjugate to $\psi$. Further, the inner product of two four-vectors is indicated by $a \cdot b = a_\mu b_\mu = \boldsymbol{a} \cdot \boldsymbol{b} - a^0 b^0$.

of coordinates $F(x'x''x''')$. Utilizing the various invariance conditions which may be placed on $F$, we may expand it in momentum space as

$$\left.\begin{aligned} F(x'x''x''') &= (2\pi)^{-8}\int d^4l_1 d^4l_3\, G(l_1 l_3) \\ &\quad \times \exp i\,[l_1(x'-x'') + l_3(x'''-x'')], \end{aligned}\right\} \quad (1.2)$$

where $G$ is a function only of $l_1^2$, $l_3^2$, and $(l_1+l_3)^2$.

Of especial importance to any investigation of a theory of this type are the questions, first, whether such a generalization of the usual theory can bring about the desired convergence of integrals representing matrix elements, and second, if it can do this, what effects this generalization would have upon such properties of the theory as its causal behaviour. Bloch(2) and Kristensen(3) have shown that, in order to gain convergence to all orders of the coupling constant, it is sufficient (and probably necessary) to require that $G(l_1 l_3)$ vanish if any of the vectors $l_1$, $l_3$ or $l_1+l_3$ is space-like. This is a rather serious restriction; in fact, it eliminates the possibility of obtaining the usual local theory as a limiting case. It has been felt that such a restriction may perhaps lead to acausal behaviour for the particles described by the theory. It is the purpose of this paper to investigate in some detail the commensurability of such an assumption with the causality requirement, and to show in what sense we may say that causality is preserved. A theory will be said to exhibit causal behaviour if it predicts that all observable signals or particles of positive energy are propagated only in a forward direction in space-time, and at a velocity equal to or less than the velocity of light.

Discussions of the application of the requirement of causality to the non-local interaction are not new, of course. For example, Bloch(2), and later Chrétien and Peierls(4), have determined what properties the form function must possess in order that the interaction be limited to a small region in space and time. In substance, their result is that, if the form function in momentum space, $G(l_1 l_3)$, is sufficiently smooth, then the interaction involves essentially only field variables at points close to each other. Smoothness here implies the continuity of the various higher derivatives of $G$ with respect to $l_1^2$, $l_3^2$, and $(l_1+l_3)^2$. This question is, however, somewhat different from that discussed by FIERZ[(5)] in his analysis of the causal behaviour of the local theory of quantum electrodynamics. It was pointed out there that, for the causality requirement to make sense, it is necessary to discuss only observable signals. This means that the predicted matrix element for some measuring process as a whole must be examined. It was shown that, if a particle (specifically, a photon) of positive energy is absorbed at an approximate distance $r$ from its point of creation, such absorption must take place at a time at least $r/c$ later than its time of emission. It is apparent that this discussion does not correspond to that given by Bloch or Chrétien and Peierls. A simple demonstration of this discrepancy is provided by a local theory in which the Feynman or causal Green's function $\Delta_F = \Delta^1 - 2i\bar{\Delta}$ is replaced by its complex conjugate $\Delta_F^*$. This would certainly satisfy the conditions of Bloch, and Chrétien and Peierls, since the interaction would only involve the field variables at the same point. Nevertheless, such a theory would not satisfy the Fierz condition, which we might call "causality in the large", because the absorption of a particle of positive energy would actually occur before its emission. In the course of our examination of the properties of the restricted non-local interaction, we shall find the distinction between these conditions appearing in a rather natural way.

Perhaps it should be mentioned that this work is rather distinct from that of VAN KAMPEN[(6)] and others, who have established rather general conditions on the $S$-matrix for scattering in classical and first-quantized theories. Their interest is mainly concentrated upon determining the properties for cross sections and bound states following from conditions which are, in a sense, weaker than those above, but which must be followed rigorously. This involves a somewhat different emphasis, resulting from a different point of view concerning the causality condition.

Essentially, it is possible to start from a given set of properties, including, say, some sort of causality condition, and from these to deduce certain characteristics which must be possessed by any theory containing these properties. This is the approach of van Kampen, mentioned above. On the other hand, it is also possible to begin with a definite theory or class of theories, and to deduce to what extent this theory possesses certain desired properties. This is the approach which has been used by Fierz, and which will be adopted here. It contains one definite advantage; namely, if we begin with a causality condition, and use this to restrict the form of the theory, then we must, of course, use a condition which can be expressed in specific terms, and which must be adhered to rigorously. However, it has been pointed out previously that such a condition in a quantum theory will tend to be rather weak, primarily because of the inability to assign precise values of the momentum and position to a particle at two different times. Therefore, it seems better for our purposes to begin with that specific theory in which we are interested, and to examine its predictions for those processes which will exhibit most clearly its causal or acausal nature. These processes are just those which describe physical methods for measuring the velocity of propagation of a particle. The more general approach, while more difficult in application, might be expected to throw considerable light on the structure of $S$-matrix theory, particularly if the same sort of causality condition as that used here could be formulated in a more definite manner. One of the problems involved in such a treatment would be the construction of certain types of localized states. We shall attempt to avoid such difficult questions by the use of a more intuitive approach.

Several basic assumptions and limitations will be introduced here in order to simplify the discussion. The most important of these involves the application of perturbation theory to the calculation of matrix elements involved in determining the causal behaviour. In particular, we shall assume that, if the results of the lowest order perturbation calculation indicate a causal behaviour, such behaviour will carry over into the higher orders. Causality will be seen to be intimately connected with the form of a certain product of the Green's function $\Delta_F$ and form functions $F(x'x''x''')$. In the higher orders, the same product is merely repeated a number of times. If this product is of the proper form to ensure causality for the first non-vanishing term in the expansion, we may expect that the higher orders will not introduce difficulties. Wherever possible, we shall attempt to indicate what modifications are introduced in the higher orders. The entire structure of our analysis might not make any sense if the perturbation method itself is not valid, of course. Such questions, while important, are not to be discussed here.

For convenience, only two types of particles will be considered, one possessing charge conjugate states and spin one-half, and one Majorana neutral particle. The fields describing the former will be denoted by $\psi$ and $\psi^+$, and $\varphi$ will be used for the field of the neutral particle. Both the neutral particle and the coupling will be assumed to be scalar. At times we may find it convenient, for giving a physical picture of the processes considered, to refer to these particles as nucleons and mesons.

Finally, we shall only be concerned with the causal or acausal behaviour of the $\varphi$ field; that is, we shall only require that the neutral particle have a velocity less than that of light. This makes it possible to treat the $\psi$ and $\psi^+$ fields non-relativistically if desirable, which can simplify the discussion. It is obvious, of course, that a similar treatment of the causality properties of the $\psi$ and $\psi^+$ fields could be given, with essentially no modification of the procedure.

## 2. Limitations on the Causality Condition.

As indicated previously, it is extremely difficult to give an exact criterion for causal behaviour of a theory, primarily due to the limitations imposed on the measuring process by the quantum nature of the theory. We shall now examine this limitation more closely. Essentially two types of measurements are involved in determining causal behaviour as defined previously. These are: the determination of the location of the particle at two different points in space-time, and the measurement of the sign of the energy of the particle. If the theory is second quantized, then of course the points of position measurement are just the points at which the creation and destruction of the particle in the given state occur.

It is rather clear that the operator measuring the position of a particle, the eigenfunctions of which are the so-called localized states of the particle, does not commute with the energy operator. This, however, is too much to expect; all we really would need for a precise formulation of the causality condition is that the position operator commute with the operator determining the sign of the particle energy. Expressed in other words, we would require that the localized states of the system be composed only of positive (or only of negative) energy components. That this should be so seems extremely unlikely, and we therefore expect that the position of the particle, or its points of creation and destruction, may only be defined to within a certain distance. This distance may be taken to be of the order of the Compton wavelength of the particle, $h/mc$, which is the position uncertainty we would obtain using states which are described by the usual minimum wave packets familiar in ordinary quantum mechanics. Furthermore, we should also expect to be able to determine the sign of the energy of the particle with only a certain probability; this probability may be large, but not equal to one. With these limitations, our statement of causal behaviour becomes as follows: to the extent to which the energy of the particle is known to be positive, and to the extent to which its points of creation and destruction may be determined, these points must be separated by a time-like distance, and the point of destruction must occur later than that of creation. At first glance, we might be tempted to require also that the particle energy be greater than $mc^2$, i.e., that the particle be real, not virtual. On the other hand, the existence of an appreciable probability for finding a virtual particle propagating at a velocity greater than $c$ at a distance from its point of creation large compared to $h/mc$ can also be considered to be a violation of causality. It seems reasonable, then, to include virtual particles in our discussion. This question does not arise in the usual local theory, for there we know that the range of the interaction produced by the exchange of a virtual particle of mass $m$ is of the order of $h/mc$, no larger than the fundamental uncertainty in the position measurement. There appears to be no reason to expect this range to be any shorter in a non-local theory. Conversely, we also may regard this as a reason for not choosing our position measurement more accurate than $h/mc$, for the existence of virtual particles prevents any more accurate formulation of the causality condition.

With these considerations in mind, we now may give a general description of the type of process the investigation of which should prove most interesting and decisive with regard to causality. We might expect such processes to be the simplest ones possible which create and subsequently destroy a meson; that is, the interaction of two nucleons by means of the meson field. The process should provide some method of determining where and when the meson is created and destroyed; this may be done by determining where the nucleons which emit and absorb the meson change state. Furthermore, the energy of the meson may be determined from a knowledge of the energy change of the nucleon involved in its emission. These requirements mean that the nucleon states must not be described by momentum eigenfunctions, but rather by some sort of wave packets, which also permit a certain localization in space and time. In principle, from a knowledge of the nucleon states in the infinite past and in the infinite future, we may deduce the properties of the particle field which transmits the interaction between the two nucleons. It does not matter whether we assume such interaction occurs by the exchange of one or many mesons; in either case the causality condition should be satisfied.

It also should be noted that, if the initial and final states of the nucleons are chosen to be free particle states, i.e., some superposition of plane waves, then an additional interaction with the meson or some other field must be introduced to provide long range (greater than $h/mc$) interaction between the nucleons. This is, of course, a consequence of the conservation of energy and momentum, which forbids the absorption or emission by free nucleons of any save virtual mesons. This additional interaction may either be with a prescribed external field, or else with the meson field or some other quantized field. In the latter case, the additional field also should be described by states which are represented by wave packets. In the next section, we shall present two types of processes which can throw light on the causal behaviour of the theory, and show that essentially the same answer would be obtained in an analysis of either of them.

## 3. Measuring Processes.

Whether or not a theory is causal can be determined from the predictions it makes for various special processes. In this section, we shall consider several representative examples of such processes, which are of the general type discussed previously. We shall show that both of these "Gedanken" experiments lead to a condition essentially the same as that of Fierz.

In the interest of simplicity, we may begin with a case in which two nucleons in different potential wells interact by means of the intermediate meson field, the dropping of one nucleon from its initial state to a lower level causing the excitation of the system containing the other nucleon to a higher state. Since the treatment of bound states in a field-theoretical manner would introduce several complications into our discussion, we shall assume that the nucleons in the potential well are not described by a quantized field, but merely by simple Schrödinger wave functions. This means that we may not use a three-point form function, but rather consider only a non-local interaction between the meson field and a source density, represented by the nucleon wave functions. The corresponding problem in electrodynamics involves the exchange of excitation of two different atoms by means of the radiation field. The approximation of treating the nucleons by Schrödinger functions is somewhat better than the familiar semi-classical radiation theory, in that the possibility of virtual-pair formation by the meson field is contained in our discussion. Effects corresponding to the radiative corrections in emission and absorption are not included, however.

The use of a potential well serves to localize the emission and absorption of the meson in space, but not in time. In order also to establish a time for these events, we may consider that the population of the nucleon states varies as a result of other unspecified interactions with other particles. This changing population results in a time-dependent normalization for the particles in each potential well:

$$\int d^3\boldsymbol{x}\, \psi^* (\boldsymbol{x}, t)\, \psi (\boldsymbol{x}, t) = \left| f(t) \right|^2, \tag{3.1}$$

and might be described by introducing an additional imaginary potential $V' = i\hbar \left[ln\, f(t)\right]'$ into the Schrödinger equation, which becomes

$$\left.\begin{aligned} \left[H^0 + V'(t)\right] \psi(t) &= i\hbar \frac{\partial \psi}{\partial t}, \\ H^0 &= -(\hbar^2/2M)\, \nabla^2 + V(\boldsymbol{x}). \end{aligned}\right\} \tag{3.2}$$

In particular, the function $f(t)$ should be appreciably different from zero only over a certain time interval.

Consider first a potential well with center at the origin. If the potential $V(\boldsymbol{x})$ is chosen to be spherically symmetric, then the solutions to (3.2) may be written as

$$\psi_{nlm} = f_n(t)\, u_n(r)\, Y_l^m(\vartheta, \varphi) \exp - i\,(E_n/\hbar)\, t, \tag{3.3}$$

in which the $u_n$ are the normalized radial parts of the energy eigenfunctions of the unperturbed Hamiltonian $H^0$. The $Y_l^m$ are chosen to be normalized so that their square integral over all angles is unity.

Now we return to our original problem of the two nucleons. Consider two different potential wells, one with center at $\boldsymbol{x}$, and with particle states which have a maximum amplitude at time $x^0$, and the other with a center at $\boldsymbol{y}$, and a maximum state amplitude at time $y^0$. We denote the wave functions of the nucleon in the first well by $\psi^1$, and those of the nucleon in the second well by $\psi^2$. Then the $S$-matrix element for a transition in which the nucleon in the first well goes from state $nlm$ to $n'l'm'$, and nucleon 2 goes from $n'l'm'$ to $nlm$, will be proportional to the integral

$$\left. \begin{aligned} I(x, y) = \int d^4x'\, d^4y'\, \overset{2*}{\psi}_{nlm}(y')\, \overset{2}{\psi}_{n'l'm'}(y') \\ \times \Delta_F'(y' - x')\, \overset{1*}{\psi}_{n'l'm'}(x')\, \overset{1}{\psi}_{nlm}(x'). \end{aligned} \right\} \tag{3.4}$$

Here, $\Delta_F'(y' - x')$ is some sort of Green's function describing the propagation of the meson from the point $x'$ to the point $y'$. It may be assumed to contain the effects of a non-local interaction between the meson field and the nucleon source density. To the lowest order in the coupling constant for the meson-nucleon interaction, $\Delta_F'(y' - x')$ becomes just a non-local modification of the usual local Green's function; we write it as

$$\Delta_F'(x) = (2\,\pi)^{-2} \int d^4k\, \Delta_F'(k) \left| g(k^2) \right|^2 \exp i\, k \cdot x, \tag{3.5}$$

with $g(k^2)$ some form factor in momentum space.

Obviously, we have

$$\left. \begin{aligned} \overset{1}{\psi}_{nlm}(x') &= \psi_{nlm}(x' - x), \\ \overset{2}{\psi}_{nlm}(y') &= \psi_{nlm}(y' - y), \end{aligned} \right\} \tag{3.6}$$

where $\psi_{nlm}(x)$ is the $nlm$ wave function for a particle in a state centered about the origin in both space and time. As usual, it is most convenient to work in momentum space. We prefer spherical polar to rectangular coordinates, both for convenience in handling spherical potentials and, more important, because the causality condition only involves the separation of events, not their relative angular orientation. Accordingly, we also introduce spherical coordinates in momentum space, writing the product of two wave functions as

$$\left.\begin{aligned}\psi^*_{nlm}(x)\,\psi_{n'l'm'}(x) &= \sum_{L,M} C^{mm';\,M}_{ll';\,L}\;Y^M_L(\vartheta,\varphi)\,(-i)\,L/\pi\\ &\quad\times\int_{-\infty}^{\infty} dk^0\,\zeta(k^0)\exp - ik^0x^0\int_0^\infty k^2dk\,j_L(kr)\,v^L_{nn'}(k),\\ \zeta(k^0) &= (2\pi)^{-\frac{1}{2}}c\int_{-\infty}^{\infty} dt\left\{\exp i\,[k^0c+(E_n-E_{n'})/\hbar]\,t\right\}f^*_n(t)\,f_{n'}(t),\\ v^L_{nn'}(k) &= (2/\pi)^{\frac{1}{2}}\int_0^\infty r^2dr\,j_L(kr)\,u^*_n(r)\,u_{n'}(r),\end{aligned}\right\}\quad(3.7)$$

and expanding the function $\Delta'_F(k)$ as

$$\Delta'_F(k) = \sum_{lm} Y^m_l(\vartheta_k,\varphi_k)\,\Delta'_{Flm}(k,k^0). \qquad (3.8)$$

Substituting (3.7) and (3.8) in (3.4), and using (3.5), we finally obtain

$$\left.\begin{aligned}I(x-y) &= 16\pi^3\sum C^{mm';\,M'}_{ll';\,L'}\,C^{m'm;\,M''}_{l'l;\,L''}\,C^{M'M''m;\,M}_{L'L''l';\,L}\,i^L\,Y^M_L(\vartheta,\varphi)\,I_{LM}(x-y),\\ I_{LM}(x-y) &= \int_0^\infty k^2dk\,j_L(kr)\int_{-\infty}^{\infty} dk^0\exp\\ &\quad - ik^0(x^0-y^0)\,\varrho(k,k^0)\,\Delta'_{Flm}(k,k^0)\end{aligned}\right\}\quad(3.9)$$

with

$$\left.\begin{aligned}\varrho(k,k^0) &= \zeta(k^0)^*\,\zeta(k^0)\,v^{L'}_{nn'}\,v^{L''}_{nn''}(k)^*\,\left|g(k^2-k^{02})\right|^2\\ r &\equiv \left|\boldsymbol{x}-\boldsymbol{y}\right|.\end{aligned}\right\}\quad(3.10)$$

The coefficients $C$ appearing in (3.7) and (3.10) are defined by

$$\left.\begin{aligned} Y_l^{m*}\,Y_{l'}^{m'} &= \sum\nolimits_{L'M'} C_{ll';\,L'}^{mm';\,M'}\,Y_{L'}^{M'} \\ Y_{L'}^{M'}\,Y_{L''}^{M''}\,Y_{l''}^{m''} &= \sum\nolimits_{LM} C_{L'L''l'';\,L}^{M'M''m'';\,M}\,Y_L^M, \end{aligned}\right\} \tag{3.11}$$

and the summation in (3.9) extends over $L$, $L'$, $L''$, $l''$, $M$, $M'$, $M''$, m$''$.

If the function f $(t)$ which limits the wave function in time is not chosen to decrease to zero too sharply, then $\varrho\,(k, k^0)$ will have a strong maximum at $k^0 = (E_{n'} - E_n)/\hbar c$. To see this, we need only recognize that $\zeta\,(k^0) = \bar{\zeta}\,[k^0 + (E_n - E_{n'})/\hbar c]$, where $\bar{\zeta}\,(k^0)$ is the Fourier integral transform of $f_n\,(t)^*\,f_{n'}\,(t)$. If the energy of the state $n$, $E_n$, is much greater than $E_{n'}$, then we may consider that a meson of positive energy $\sim -\hbar k^0/c$ is created at or about $x$, and destroyed near $y$. Therefore, the causality condition requires that, if $\varrho\,(k, k^0)$ is different from zero essentially only for $k^0 < 0$, $I_{LM}\,(x-y)$ should be different from zero only if $y$ is essentially within or on the forward light cone of $x$. This is just the sort of condition Fierz obtains.

Of course, the better we define the time of the meson creation or destruction, the less well-defined is the energy $-\hbar k^0/c$. The extent of the uncertainty in our condition may be estimated by choosing a particular form for $f\,(t)$, for example, a Gaussian in time:

$$f\,(t) = \exp -\frac{1}{2}\gamma^2 t^2. \tag{3.12}$$

Then, $\zeta\,(k^0)$ becomes

$$\zeta\,(k^0) = (^1/_2\,\gamma)\,\exp -\frac{1}{4}\gamma^{-2}\,[k^0 + (E_n - E_{n'})/\hbar c]^2, \tag{3.13}$$

which is also a Gaussian function. Assuming that $E_n - E_{n'}$ is much greater than $mc^2$, the meson energy is fairly well defined as positive if $\gamma \sim mc^2$. With this value of $\gamma$, the uncertainty in the time at which the meson is created, as measured by the width of the maximum in $f\,(t)^*\,f\,(t)$, is of the order of $\hbar/mc^2$. Thus, the causality condition cannot restrict the propagation properties of the meson to within a distance any smaller than $\sim c\hbar/mc^2 = \hbar/mc$. This is a rather reasonable result, since we frequently think of the Compton wavelength as some sort of extension of the meson.

The uncertainty in spatial location of the points of meson creation and destruction does not play a role in the above discussion, since this may in principle be reduced indefinitely by decreasing the range and increasing the depth of the potential well. For example, for a squarewell potential, of range $R$ and depth $V_0$, the nucleon wave function for energy $E_n$ is appreciable only for $r < R + \sqrt{-\hbar^2/2\,ME_n}$. For an $s$-state, this distance is much less than the meson Compton wavelength if $V_0 \gg (m/M)\,mc^2$.

Although the previous measuring process contains the essential elements necessary to ascertain the causal or acausal behaviour, a major objection may be raised to it. This is that the nucleon was not described by a quantized field, but rather was assumed to obey a non-relativistic Schrödinger equation. The main reason for doing this was that we wished to consider nucleons in bound states, but still avoid some of the difficulties which occur in present-day treatments of bound-state problems. Particular difficulties may be encountered in applying theories with non-local interaction to bound states.(7) On the other hand, our principal goal is to investigate the properties of a non-local interaction between two quantized fields, thus replacing one of the fields by an effective "source distribution", for the other field certainly limits the scope of our discussion.

Instead of arguing, as previously, that the non-local effects may be described completely by an altered meson Green's function, we may propose a second process in which both nucleon and meson are treated as quantized fields. Accordingly, a scattering problem involving nucleons in states of energy greater than $Mc^2$ will now be considered. As remarked previously, it is necessary to introduce an additional interaction to permit the emission and absorption of non-virtual mesons. We shall choose this additional interaction to be with the electromagnetic field, thus involving only the nucleons and not the neutral mesons. The electromagnetic field will not be treated as an external field, but rather as being quantized according to the usual theory. It is necessary, however, to assume that this nucleon-photon interaction is local, to avoid difficulties with both the gauge invariance and the construction of the $S$-matrix.

The particular measuring experiment is illustrated schematically in Fig. 1. We consider that initially we have two nucleons and one photon, each described by wave packets containing only positive frequencies. The packets of nucleon 1 and the photon appear to intersect ("collide") in a region $R_1$ in space and time. The packet of the other nucleon, nucleon 2, passes through a second region, $R_2$, well separated from $R_1$. We

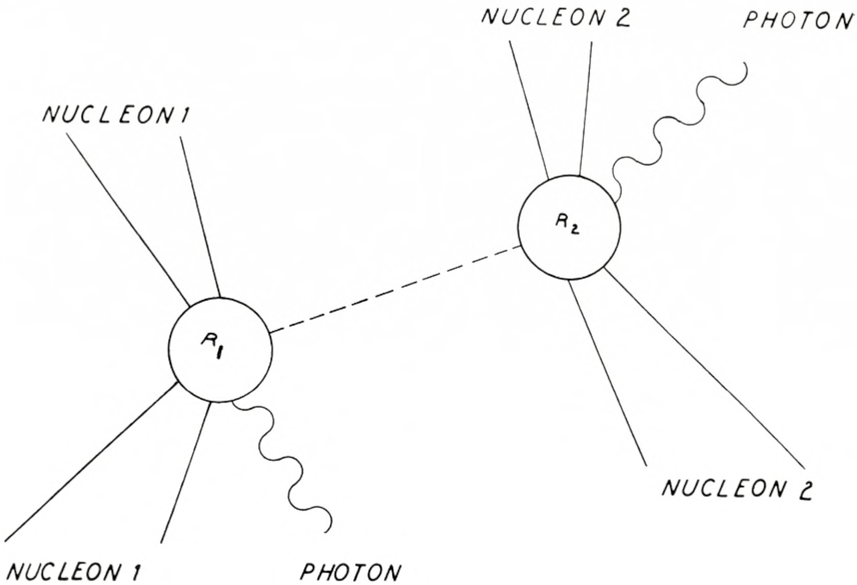


Fig. 1.

look for transitions to a final state in which we again have two nucleons and a photon, but with nucleon 2 and the photon now coming from $R_2$, and nucleon 1 from $R_1$. The interpretation is then that nucleon 1 has absorbed the photon, transferring this energy to the other nucleon by the exchange of a meson, the energy of this meson finally appearing in the photon emitted by nucleon 2. If all the wave packets are chosen to be minimum packets in either $R_1$ or $R_2$, and if the photons and corresponding nucleons do not have approximately the same direction of motion, then the photon absorption and emission must take place in and around $R_1$ and $R_2$, respectively. An analysis of the Compton effect shows then that the nucleon must lose its excitation energy by meson or photon emission within a region of dimensions of the order of magnitude of $\hbar/Mc$. This is also ensured if the initial and final state wave packets of nucleons

1 and 2 are chosen so that they only overlap in regions $R_1$ and $R_2$, respectively. Then to the extent to which the regions of intersection, $R_1$ and $R_2$, are well defined, the emission of the meson and its subsequent absorption occur in $R_1$ and $R_2$. But if the energies of both the photons are positive, then the meson going from $R_1$ to $R_2$ must have a positive energy, and our causality condition requires that the region $R_2$ must lie on or within the forward light cone of $R_1$.

Now let us turn to a description of the process by our field theory. We choose as the action $I$

$$I = \int d^4x\, \mathcal{L}_0(x) + \int d^4x\, \mathcal{L}_{EM}(x) + \int d(123)\, \mathcal{L}_M(123), \qquad (3.14)$$

where $\mathcal{L}_0$ is the usual free-field Lagrangian density for a system of mesons, nucleons, and the electromagnetic field, described by operators $\varphi$; $\psi$, $\psi^+$; $A_\mu$, respectively. The meson-nucleon interaction density is taken as

$$\mathcal{L}_M(123) = -g/2\,[\psi^+(1)\,\varphi(2)\,\psi(3) - \tilde{\psi}(1)\,\varphi(2)\,\tilde{\psi}^+(3)]\,F(123), \quad (3.15)$$

and the interaction of the nucleon with the electromagnetic field is described by

$$\mathcal{L}_{EM}(x) = ie/2\,[\psi^+(x)\,\gamma_\mu A_\mu(x)\,\psi(x) - \tilde{\psi}(x)\,A_\mu\tilde{\gamma}_\mu\tilde{\psi}^+(x)]. \quad (3.16)$$

A perturbation expansion for the $S$-matrix element for the process may be found by introducing a type of interaction representation, in which only the nucleon-photon interaction is chosen for $H_{\text{int}}$. That is, the state vector in our interaction representation, $\Psi_I$, is related to that vector in the Heisenberg representation with which it coincides at $t_0$, $\Psi_H$, by

$$\Psi_I(t) = P\left[\exp - i\int_{t_0}^{t} dt' \int d^3x' H_{EM}(x')\right] \Psi_H, \qquad (3.17)$$

with $P$ denoting the time-ordered product. We put $\hbar = c = 1$. The $S$-matrix expressed in terms of operators in this representation may be found by the method of KÄLLÉN(8) and YANG and FELDMAN(9). It is then possible to write down the $S$-matrix in the Heisenberg representation, remembering that the Green's function transforms as the product of two field operators at different space-time points. In the perturbation expansion, the lowest order terms which have non-vanishing matrix elements

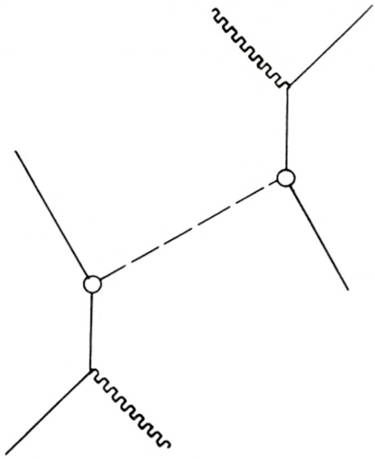

Fig. 2.

between the states considered (each with two nucleons and one photon) will be of order $e^2$ or $e^2g^2$ in the coupling constants. The terms of order $e^2g^2$ in the matrix element are of two types,

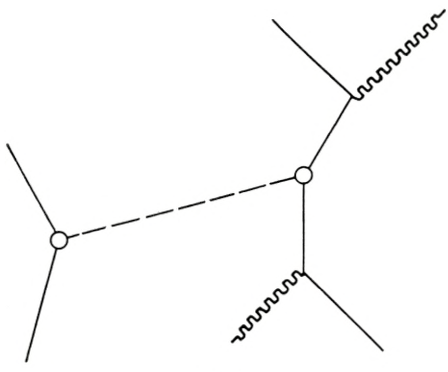

Fig. 3.

depending on whether the two photons interact with different or with the same nucleon. Sample graphs corresponding to these two types are shown in Figs. 2 and 3. The other graphs differ only in regard to which one or two of the particular nucleon line or lines the photon line is attached. An exception to this are the disconnected graphs, which correspond to no meson exchange; these give a contribution only because the initial and final states chosen for a particular nucleon are not orthogonal. These contributions, which include all those from terms of order $e^2$, will be essentially negligible if the change in the mean momentum of one of the nucleons is large compared to the spread of momenta in the nucleon wave packet. Similarly, terms corresponding to graphs of the type shown in Fig. 3 refer to processes in which the meson involved is virtual. Contributions from these may also be shown to be negligible unless $R_1$ and $R_2$ are separated by a distance less than $\hbar/mc$. In fact, with the restricted type of form factor in which we are particularly interested, these terms are identically zero. We are thus left only with graphs such as shown in Fig. 2. One part of the matrix element for the particular graph illustrated is

$$\left.\begin{aligned} I = e^2 g^2/8 \int d\,(1 \ldots 8)\, F\,(123)\, F\,(456)\, \psi_d^+\,(7)\, A_{\nu f}\,(7)\, \gamma_\nu \bar{S}\,(7-1) \\ \times \Delta_F\,(2-5)\, \psi_b\,(3)\, \psi_c^+\,(4)\, \bar{S}\,(6-8)\, A_{\mu e}\,(8)\, \gamma_\mu \psi_a\,(8), \end{aligned}\right\} \quad (3.18)$$

in which $\psi_a$ and $\psi_b$ are the initial state wave functions of nucleons 1 and 2, and $\psi_c$ and $\psi_d$ are the final state wave functions. The other parts differ only by permutations of the initial and final state wave functions. The initial and final state potentials of the electromagnetic field are denoted by $A_{\mu e}$ and $A_{\nu f}$. Here again $\Delta_F\,(x)$ is the Feynman Green's function for the meson field. The essential propagation properties of the meson field are rooted in $\Delta_F$ and in the form factors.

The wave functions $\psi_a$, $\psi_c$, and $A_{\mu e}$ refer to particles which pass through region $R_1$, whereas $\psi_b$, $\psi_d$, and $A_{\nu f}$ describe particles passing through $R_2$. If we denote by $x_1$ the midpoint of the region $R_1$, and by $x_2$ the midpoint of $R_2$, then we may define new translated wave functions $\psi'$ by the conditions

$$\left.\begin{aligned} \psi_a'\,(x) &\equiv \psi_a\,(x + x_1) & \psi_e'\,(x) &\equiv \psi_b\,(x + x_2) \\ \psi_c'\,(x) &\equiv \psi_c\,(x + x_1) & \psi_d'\,(x) &\equiv \psi_d\,(x + x_2) \\ A_{\mu e}'\,(x) &\equiv A_{\mu e}'\,(x + x_1) & A_{\nu f}'\,(x) &\equiv A_{\nu f}'\,(x + x_2). \end{aligned}\right\} \quad (3.19)$$

Then, the primed wave functions should all be in the form of packets passing through the origin; that is, at time $t = 0$ they should be minimum packets with center at $x = 0$. The matrix element (3.18) may be written more simply in terms of the Fourier transforms of the wave packets and Green's functions. The wave functions are expanded as

$$\left.\begin{aligned} \psi_a'(x) &= (2\pi)^{-2}\int d^4k\, v_a(k)\,\frac{1+e(k)}{2}\,\delta(k^2+M^2)\exp ik\cdot x,\\ A_{\mu e}'(x) &= (2\pi)^{-2}\sum_{r=1,2}\int d^4k\, N_\mu^r(k)\, a_{re}(k)\,\frac{1+e(k)}{2}\,\delta(k^2)\exp ik\cdot x, \end{aligned}\right\} \quad (3.20)$$

with similar formulas holding for the other functions. Here, $N_\mu^r(k)$ is a unit vector in the direction of polarization $r$, and, as a consequence of the supplementary condition on the potentials,

$$k_\mu N_\mu^r(k) = 0, \quad r = 1,2, \tag{3.21}$$

for transverse polarizations $r = 1,2$. The functions $v_a$ must satisfy

$$\delta(k^2+M^2)(\gamma_\mu k_\mu + iM)\, v_a = 0. \tag{3.22}$$

Using the expansions (1.2) and (3.20) for the form function and the wave functions, we obtain

$$I = e^2g^2/8\int d^4k\, M_1(k)\, M_2(k)\, \Delta_F(k)\exp ik\cdot(x_2 - x_1), \tag{3.23}$$

with

$$\left.\begin{aligned} M_1(k) &= (2\pi)^{-3}\int d^4k_1\, d^4\varkappa_1 \sum_s N_\mu^s\, v_c^+(k_1+\varkappa_1-k)\,\gamma_\mu\,\frac{(k_1+\varkappa_1+iM)}{(k_1+\varkappa_1)^2+M^2}\\ &\times v_a(k_1)\, a_{se}(\varkappa_1)\,\frac{1+e(k_1)}{2}\,\frac{1+e(\varkappa_1)}{2}\,\frac{1+e(k_1+\varkappa_1-k)}{2}\\ &\times \delta(k_1^2+M^2)\,\delta(\varkappa_1)^2\,\delta[k_1+\varkappa_1-k)^2+M^2]\, G(k_1+\varkappa_1-k, -k_1-\varkappa_1); \end{aligned}\right\} \quad (3.24)$$

$$\left.\begin{aligned} M_2(k) &= (2\pi)^{-3}\int d^4k_2\, d^4\varkappa_2 \sum_r N_\nu^r\, v_d^+(k_2)\,\gamma_\nu\,\frac{(k_2+\varkappa_2+iM)}{(k_2+\varkappa_2)^2+M^2}\\ &\times v_b(k_2+\varkappa_2-k)\, a_{rf}(\varkappa_2)\,\frac{1+e(k_2)}{2}\,\frac{1+e(\varkappa_2)}{2}\,\frac{1+e(k_2-k+\varkappa_2)}{2}\\ &\times \delta(k_2^2+M^2)\,\delta(\varkappa_2^2)\,\delta[(k_2-k+\varkappa_2)^2+M^2]\, G(k_2+\varkappa_2-k_2, -\varkappa_2+k); \end{aligned}\right\} \quad (3.25)$$

and

$$\Delta_F(k) = -2i(2\pi)^{-2}[k^2+M^2-i\epsilon]^{-1}. \tag{3.26}$$

In general, the form of $M_1(k)$ and $M_2(k)$ will depend upon the particular choice of the form factor in momentum space, $G(l_1, l_3)$, and on the form of the wave packets selected. However, two properties of considerable importance for our purposes may be deduced without further specialization. The first of these is that

$$M_1(k) = M_2(k) = 0 \qquad \text{for} \left.\begin{matrix} k^2 < 0 \\ k^0 < 0, \end{matrix}\right\} \tag{3.27}$$

which means that only the positive frequency components of $\Delta_F(k)$ need enter into our analysis. Of course, (3.27) does not eliminate contributions from space-like vectors $k$ with $k^0 < 0$, but such vectors may all be transformed into vectors with positive frequency components by proper Lorentz transformations. We shall in fact later require that the propagation Green's function be such that the virtual particles described by $k^2 > 0$ give only short-range effects.

Consider the definition (3.24) for $M_1(k)$. The integrals contain a factor

$$\left.\begin{aligned} &\delta(k_1^2 + M^2)\,\delta(\varkappa_1^2)\,\delta[(k_1+\varkappa_1-k)^2+M^2]\frac{1+\epsilon(k_1)}{2}\frac{1+\epsilon(\varkappa_1)}{2}\frac{1+\epsilon(k_1+\varkappa_1-k)}{2} \\ &\quad = \delta(k_1^2 + M^2)\,\delta(\varkappa_1^2)\,\delta\,[k^2 + 2\,k_1\cdot\varkappa_1 - 2\,k\cdot(k_1+\varkappa_1)] \\ &\qquad \times \frac{1+\epsilon(k_1)}{2}\frac{1+\epsilon(\varkappa_1)}{2}\frac{1+\epsilon(k_1+\varkappa_1-k)}{2}. \end{aligned}\right\} \tag{3.28}$$

But if $a$ and $b$ are two time-like vectors, $a\cdot b$ is positive if $a^0$ and $b^0$ are of opposite signs, and negative if they are of the same sign. Hence,

$$k^2 + 2\,k_1\cdot\varkappa_1 - 2\,k\cdot(k_1+\varkappa_1) < 0 \tag{3.29}$$

for all vectors $k$ such that $k^2 < 0$, $k^0 < 0$. Thus, for these vectors the $\delta$-function is always zero, and the integral vanishes identically. A similar argument holds for $M_2(k)$. Therefore, our integration in (3.23) only need go over space-like vectors $k$, and over time-like vectors with $k^0 > 0$.

The second general property of $M_1(k)$ and $M_2(k)$ deals with their smoothness when considered as functions of the vector $k$; that is, the continuity of their derivatives of a given order with respect to $k$. It may be shown that, if $G(l_1, l_3)$ and the various functions $v(k)$, $a_r(\varkappa)$ are sufficiently smooth, and $n$ is any finite positive integer, then the derivatives of $M_1(k)$ and $M_2(k)$ with respect to $k$ and of $n$th order are continuous everywhere, except possibly at $k^2 = 0$. This is not a completely trivial property, for the presence of the product of the various $\delta$-functions might be thought to introduce discontinuities in some higher order. For example, the integral

$$I(\alpha) = \int_0^1 dx \int_0^1 dy\, \delta(x + y - \alpha) \tag{3.30}$$

does not possess a continuous first derivative $I'(\alpha)$ everywhere. In our case, the integral $M_1(k)$ is actually an integral over a five-dimensional surface embedded in the eight-dimensional space spanned by $k_{1\mu}$, $\varkappa_{1\mu}$. This surface is formed by the intersection of the surfaces

$$\left.\begin{aligned} k_1^2 + M^2 = 0, \quad \varkappa_1^2 = 0, \\ (k_1 + \varkappa_1 - k)^2 + M^2 = 0, \end{aligned}\right\} \tag{3.31}$$

and depends upon $k$ as a parameter. The desired smoothness results from the fact that the vectors $k_{1\mu}$ and $\varkappa_{1\mu}$ depend upon the five independent variables of the surface and on the parameter $k$ in a continuous manner, a condition which is not met for (3.30). For the proof, it is necessary, among other points, to show that the equations (3.31) have a solution for all values of $k^2 > 0$ and of $k^2 < 0$, $k^0 > 0$. This means that mesons of all momenta are to be involved in the matrix element (3.23).

With these properties in mind, and with reference to the matrix element (3.23), we see that our causality condition takes a particularly simple form. It is: if $\varrho(k) \equiv M_1(k)\, M_2(k)$ vanishes for $k^2 < 0$, $k^0 < 0$, and possesses only discontinuities in its derivatives corresponding to those of the form factors entering into its definition, then the integral

$$I = e^2 g^2/8 \int d^4k\, \varrho(k)\, \Delta_F(k) \exp i\, k \cdot (x_2 - x_1)$$

must be essentially different from zero only if $x_2$ is on or within the forward light cone of $x_1$. The relation of this condition to that obtained from our previous measuring process, expressed in (3.9), is now apparent.

We are fairly sure of the validity of the use of a perturbation expansion to describe the interaction of the nucleon with the electromagnetic field, but it is a much more doubtful technique for treating the meson-nucleon interaction. It would certainly be desirable to know the effect of terms of higher order in $g^2$ on the matrix element, in the very least. Some of these terms will refer to processes such as the creation and annihilation of virtual nucleon-anti-nucleon pairs by the meson field. Neglecting the possibility of an interaction of these nucleons with the electromagnetic field, these pairs may presumably be removed by some sort of renormalization. In any case, their only effect will be to modify somewhat the propagation function $\Delta_F(k)$ appearing in (3.23). Since it introduces no more difficulty, we shall henceforth anticipate this modification, and replace $\Delta_F(k)$ by some effective Green's function $\Delta'_F(k)$. Other terms will refer to nucleon self-energy effects and may involve the electromagnetic field in a rather complicated manner. Nevertheless, it is easy to see that such effects do not in essence change the argument. However, one type of term which is definitely not included in our considerations is the meson analogue of the radiative corrections to scattering. These essentially replace the meson-nucleon vertices in Figs. 2 and 3 by some complicated vertex parts. We shall not discuss the effects of such processes here, save to remark that in a certain sense, for our purposes, they may be equivalent to modifying the form factor $F(123)$ somewhat. Whether or not they affect the causality properties depends to a certain extent upon the conditions which we obtain for $F(123)$.

## 4. Asymptotic Expansion of the Integral.

We have seen that our causality condition requires a knowledge of the behaviour of the integral

$$I(x) = \int d^4k\, \varrho(k)\, \Delta'_F(k) \exp i\, k\cdot x \tag{4.1}$$

for $|\boldsymbol{x}| \gg 1/m$. In this section, we shall determine the properties of this integral in terms of the properties of $\Delta'_F(k)$ and the function $\varrho(k)$. It should be noted first that the singularities of the integrand can be of two major types. First, the function $\Delta'_F(k)$ may introduce either poles or distributed singularities culminating in branch points. Thus the zeroth order term in a perturbation expansion of $\Delta'_F$, which is just Feynman's $\Delta_F$, has simple poles at $k^0 = \pm\sqrt{\boldsymbol{k}^2 + m^2 \mp i\epsilon}, \epsilon > 0$. Furthermore, the function $\varrho(k)$ may have discontinuities in either itself, or in its derivatives. We introduce here the requirement that $G(l_1, l_3)$ may possess such discontinuities only along the surfaces $l_1^2 = 0$, $l_3^2 = 0$, or $(l_1 + l_3)^2 = 0$. Then the discontinuities of $\varrho(k)$ will be limited to the surface $k^2 = 0$.

It is possible to separate these two types of singularities into different terms. For example, the function $\varrho(k)(k^2 + m^2 - i\epsilon)^{-1}$ may be written as

$$\left.\begin{aligned}\frac{\varrho(\boldsymbol{k}, k^0)}{k^2+m^2-i\epsilon} = &\left[\frac{\varrho(\boldsymbol{k}, k^0)}{k^2+m^2-i\epsilon} - \frac{\varrho(\boldsymbol{k}, k^0_-)}{(k^0-k^0_-)(k^0_+-k^0_-)} - \frac{\varrho(\boldsymbol{k}, k^0_+)}{(k^0-k^0_+)(k^0_- - k^0_+)}\right] \\ &+ \frac{\varrho(\boldsymbol{k}, k^0_-)}{(k^0-k^0_-)(k^0_+-k^0_-)} + \frac{\varrho(\boldsymbol{k}, k^0)}{(k^0-k^0_+)(k^0_- - k^0_+)}\end{aligned}\right\} \quad (4.2)$$

where

$$k^0_\pm = \pm\sqrt{\boldsymbol{k}^2 + m^2 \mp i\epsilon}. \quad (4.3)$$

Then the term in brackets in (4.2) no longer possesses the poles at $k^0 = k^0_+$ or $k^0_-$, while the second and third terms do not have the discontinuities of $\varrho(k)$. Distributed singularities may be handled in the same manner, save that now the coefficients of the subtracted terms should be otherwise analytic functions which coincide with $\varrho(k)\Delta'_F(k)$ along the branch cut. After this is performed, the function $\varrho(k)\Delta'_F(k)$ may be written as the sum of two functions, $f(k)$ and $g(k)$, in which $f(k)$ has no singular points other than the discontinuities of $\varrho(k)$, and $g(k)$ has only poles and branch points corresponding to those of $\Delta'_F(k)$. Furthermore, $g(k)$ can have no singularities in the region $k^2 < 0$, $k^0 < 0$, for here $\varrho(k) \equiv 0$, and the coefficients of the subtracted terms must be zero. We now may consider the Fourier transforms of the two functions $f(k)$ and $g(k)$ separately.

Consider first the integral

$$I^1(x) = (2\pi)^{-2}\int d^4k\, f(k) \exp i\, k\cdot x. \tag{4.4}$$

It is convenient to expand $I^1(x)$ in spherical harmonics, obtaining

$$I^1(x) = i/(2\pi^3) \sum_{lm} i^l Y_l^m(\vartheta, \varphi)\, I_{lm}^1(r, x^0), \tag{4.5}$$

where

$$\left.\begin{aligned} I_{lm}^1(r, x^0) &= \int_0^\infty dk\, k^{l+1} j_l(kr) \int_{-\infty}^{\infty} dk^0 f_{lm}(k, k^0) \exp - ik^0x^0, \\ f(k) &= \sum_{lm} Y_l^m(\vartheta_k, \varphi_k)\, f_{lm}(k, k^0)\, k^{l-1}. \end{aligned}\right\} \tag{4.6}$$

An asymptotic expansion for $I_{lm}^1(r, x^0)$ may be obtained by a method which is a slight generalization of that given by Willis[(10)]. If $f_{lm}(k, k^0)$ is square integrable, then we may write

$$\left.\begin{aligned} I_{lm}^1(r, x^0) = \underset{\alpha_1\to 0}{L}\ \underset{\alpha_2\to 0}{L} &\int_0^\infty dk\, k^{l+1} j_l(kr) \exp - \alpha_1 k \\ &\times \int_{-\infty}^{\infty} dk^0 f_{lm}(k, k^0) \exp\left[- ik^0x^0 - \alpha_2 |k^0|\right]. \end{aligned}\right\} \tag{4.7}$$

We know that, save on the surface $\boldsymbol{k}^2 = k^{02}$, $f(k, k^0)$ possesses continuous derivatives of order $N+1$, for any finite $N$. Then we have

$$\left.\begin{aligned} f_{lm}(k, k^0) &= \sum_{n=0}^{N-1} \frac{1}{n!} \sum_{\nu=0}^{n} \binom{n}{\nu} f_{+}^{n-\nu\,\nu}(0{,}0)\, k^{n-\nu} k^{0\nu} + R_{+}^N, \quad k^2 > k^{02}; \\ &= \sum_{n=0}^{N-1} \frac{1}{n!} \sum_{\nu=0}^{n} \binom{n}{\nu} f_{-+}^{n-\nu\,\nu}(0{,}0)\, k^{n-\nu} k^{0\nu} + R_{-+}^N, \quad k^2 < k^{02}; \ k^0 > 0; \\ &= \sum_{n=0}^{N-1} \frac{1}{n!} \sum_{\nu=0}^{n} \binom{n}{\nu} f_{--}^{n-\nu\,\nu}(0{,}0)\, k^{n-0} k^{0\nu} + R_{--}^N, \quad k^2 < k^{02}; \ k^0 < 0; \end{aligned}\right\} \tag{4.8}$$

where

$$\left.\begin{aligned} f_{+}^{n-\nu\,\nu}(0{,}0) &= \underset{k^2\to k^{02}+}{L}\ \underset{k\to 0}{L}\ \frac{\partial^{n-\nu}}{\partial k^{n-\nu}} \frac{\partial^\nu}{\partial k^{0\nu}} f(k, k^0), \\ f_{-+}^{n-\nu\,\nu}(0{,}0) &= \underset{k^2\to k^{02}-}{L}\ \underset{k^0\to 0+}{L}\ \frac{\partial^{n-\nu}}{\partial k^{n-\nu}} \frac{\partial^\nu}{\partial k^{0\nu}} f(k, k^0), \\ f_{--}^{n-\nu\,\nu}(0{,}0) &= \underset{k^2\to k^{02}-}{L}\ \underset{k^0\to 0-}{L}\ \frac{\partial^{n-\nu}}{\partial k^{n-\nu}} \frac{\partial^\nu}{\partial k^{0\nu}} f(k, k^0). \end{aligned}\right\} \tag{4.9}$$

The use of the subscript plus and minus signs is in cognizance of the fact that the derivatives of the function $f(k, k^0)$ may be discontinuous across the surface $\boldsymbol{k}^2 \equiv k^{02}$. The remainders in (4.8) may be written as

$$R^N = \frac{1}{N!} \sum_{\nu=0}^{N}{}' \binom{N}{\nu} f^{N-\nu\,\nu}(\beta k, \beta k^0)\, k^{N-\nu}\, k^{0\,\nu},\ 0 < \beta < 1. \tag{4.10}$$

If we introduce

$$\left.\begin{aligned}
\varphi_{+}\ \ (\alpha_1\alpha_2) &= \int_0^\infty dk\, k^{l+1}\, j_l(kr) \exp - \alpha_1 k \int_{-k}^{k} dk^0 \exp\left[-ik^0x^0 - \alpha_2 \left| k^0 \right|\right],\\
\varphi_{-+}(\alpha_1\alpha_2) &= \int_0^\infty dk\, k^{l+1}\, j_l(kr) \exp - \alpha_1 k \int_{k}^{\infty} dk^0 \exp\left[-ik^0x^0 - \alpha_2 k^0\right],\\
\varphi_{--}(\alpha_1\alpha_2) &= \int_0^\infty dk\, k^{l+1}\, j_l(kr) \exp - \alpha_1 k \int_{\infty}^{-k} dk^0 \exp\left[-ik^0x^0 + \alpha_2 k^0\right];
\end{aligned}\right\} \tag{4.11}$$

and denote $\dfrac{\partial^{n-\nu}}{\partial {\alpha_1}^{n-\nu}} \dfrac{\partial^{\nu}}{\partial {\alpha_2}^{\nu}} \varphi(\alpha_1\alpha_2)$ by $\varphi^{n-\nu\,\nu}(\alpha_1\alpha_2)$,

we have

$$\left.\begin{aligned}
I^1_{lm}(r, x^0) = \underset{\alpha_1\to 0}{L}\ \underset{\alpha_2\to 0}{L} \Bigg\{ &\sum_{n=0}^{N-1}{}' \frac{(-)^n}{n!} \sum_{\nu=0}^{n}{}' \binom{n}{\nu} \left[f_{+}^{n-\nu\,\nu}(0{,}0)\, \varphi_{+}^{n-\nu\,\nu}(\alpha_1\alpha_2)\right.\\
&\left. + f_{-+}^{n-\nu\,\nu}(0{,}0\, \varphi_{-+}^{n-\nu\,\nu}(\alpha_1\alpha_2) + f_{--}^{n-\nu\,\nu}(0{,}0)\, \varphi_{--}^{n-\nu\,\nu}(\alpha_1\alpha_2)\right] + R_N \Bigg\}.
\end{aligned}\right\} \tag{4.12}$$

The remainder here is given by

$$R^N = \sum_{\nu=0}^{N} \left[C_{+}\, \varphi_{+}^{N-\nu\,\nu}(\alpha_1\alpha_2) + C_{-+}\, \varphi_{-+}^{N-\nu\,\nu}(\alpha_1\alpha_2) + C_{--}\, \varphi_{--}^{N-\nu\,\nu}(\alpha_1\alpha_2)\right]; \tag{4.13}$$

the coefficients $C$ may be shown to be finite if $f^{N-\nu\,\nu}$ is of bounded variation. We shall henceforth make this assumption. All that remains in order to obtain an asymptotic expansion for $I^1_{lm}(r, x^0)$ is to evaluate the coefficients $\varphi^{n-\nu\,\nu}(\alpha_1\alpha_2)$. In general, this can be done only in terms of an infinite series in either $r/x^0$ or $x^0/r$, according as $r$ is less than or greater than $\left|x^0\right|$. We find, for $r > \left|x^0\right|$,

$$\left.\begin{aligned}
\varphi_{-+}^{n-\nu\,\nu} &= 2^{l+1}(-i)^{n+1}\, r^{-n-l-3} + S_{n\nu},\\
\varphi_{--}^{n-\nu\,\nu} &= 2^{l+1}(+i)^{n+1}(-)^{\nu}\, r^{-n-l-3} + S_{n\nu},\\
\varphi_{+}^{n-\nu\,\nu}(\alpha_1\alpha_2) &= -\varphi_{-+}^{n-\nu\,\nu}(\alpha_1\alpha_2) - \varphi_{--}^{n-\nu\,\nu}(\alpha_1\alpha_2),\\
S_{n\nu} &\equiv \sum_{\sigma \geq \frac{1}{2}(n+1)} \frac{(l+\sigma)!\,(2\sigma)!}{(2\sigma-n+\nu)\,\sigma!\,(2\sigma-n-1)!}(x^0/r)^{2\sigma-n-1},
\end{aligned}\right\} \quad (4.14)$$

in which only terms of order $\alpha_1^0$, $\alpha_2^0$ have been retained. If $I_{lm}^1$ is rewritten as

$$\left.\begin{aligned}
I_{lm}^1(r, x^0) = \underset{\alpha_1 \to 0}{L}\ \underset{\alpha_2 \to 0}{L} \sum_{n=0}^{N-1} \frac{(-)^n}{n!} \sum_{\nu=0}^{n} \binom{n}{\nu} \{\varphi_{-+}^{n-\nu\,\nu}(\alpha_1\alpha_2)\,[f_{-+}^{n-\nu\,\nu}(0,0)\\
- f_{+}^{n-\nu\,\nu}(0,0)] + \varphi_{--}^{n-\nu\,\nu}(\alpha_1\alpha_2)\,[f_{--}^{n-\nu\,\nu}(0,0) - f_{+}^{n-\nu\,\nu}(0,0)]\} + R_N,
\end{aligned}\right\} \quad (4.15)$$

we see that $I_{lm}^1(r, x^0)$ decreases in a space-like direction more rapidly than $r^{-n-l-3}$ if $f^{n-\nu\nu}(k, k^0)$ is continuous across the surface $k^2 = k^{02}$. The coefficient of the term of order $r^{-n}$ in an asymptotic expansion is thus of the order of magnitude of the discontinuity in the $n$-4th derivative of $\varrho(k)$. This agrees with a simple calculation of the effect of a discontinuous form factor upon the propagation of signals.

Similarly, if $|x^0| > r$, we have

$$\left.\begin{aligned}
\varphi_{-+}^{n-\nu\,\nu} &= 2^{l+1}(-)^{l}(+i)^{n+1}\, x^{0\,-n-l-3}(r/x^0)^l S_{n\nu},\\
\varphi_{--}^{n-\nu\,\nu} &= 2^{l+1}(-)^{l-\nu}(-i)^{n+1}\, x^{0\,-n-l-3}(r/x^0)^l S_{n\nu},\\
\varphi_{+}^{n-\nu\,\nu}(\alpha_1\alpha_2) &= -\varphi_{-+}^{n-\nu\,\nu}(\alpha_1\alpha_2) - \varphi_{--}^{n-\nu\,\nu}(\alpha_1\alpha_2),\\
S_{n\nu} &\equiv \sum_{\sigma=0}^{\infty} \frac{(l+\sigma)!\,(2\sigma+2l+n+1)!}{(2\sigma+2l+n-\nu+2)(\sigma)!\,(2\sigma+2l+1)!}(r/x^0)^{2\sigma}.
\end{aligned}\right\} \quad (4.16)$$

In this case, $I_{lm}^1(r, x^0)$ decreases in a time-like direction more rapidly than $x^{0\,-n-l-3}$ if $f^{n-\nu\nu}(k, k^0)$ is continuous across the light cone.

The case of $r = |x^0|$, that is, on the light cone itself, requires special attention. For $r = |x^0|$, neither the infinite series in (4.14), nor that in (4.16), converges, and hence our method for obtaining the asymptotic expansion breaks down. We might argue on physical grounds that the indeterminacy of the behaviour on the exact surface $r = x^0$, which is of zero measure, should introduce no difficulty. A more careful analysis, however, requires that we examine the behaviour of the integral of $I(r, x^0)$ taken over some small volume element spanning the light cone, in the limit in which this element is located far from the origin. If this is done, we see immediately that at worst the decrease with distance from the origin goes only as $r^{-n-2}$, for functions with discontinuities in the $n$th derivative. Thus no real problem is presented by this singular case.

The continuity of the derivatives of $\varrho(k)$, and hence of $f(\boldsymbol{k}, k^0)$, may be related to the continuity of the derivatives of $G(l_1, l_3)$ with respect to $(l_1 + l_3)^2 = k^2$. We are particularly interested in the case where $G(l_1, l_3)$ vanishes for $(l_1 + l_3)^2$ greater than zero. For this type of form factor, $\varrho(k)$ will have derivatives of order $2n$ continuous across the surface $k^2 = 0$, if $G(l_1, l_3)$ has derivatives with respect to $(l_1 + l_3)^2$ of order $n$ which are continuous across $(l_1 + l_3)^2 = 0$. The factor two arises from the fact that $\varrho(k)$ contains the product of two form factors. The discontinuities in the derivatives of the other factors in $M_1(k)$ and $M_2(k)$ will play no part if the first $n$ derivatives of $G(l_1, l_3)$ with respect to $(l_1 + l_3)^2$ are zero at $(l_1 + l_3)^2 = 0$, as they must be if $G$ is to vanish identically for $l_1 + l_3$ space-like.

Thus far we have been concerned only with the integral $I^1$. The discussion of the Fourier transform of $g(k)$,

$$I^2(x) = (2\pi)^{-2} \int d^4k \, g(k) \exp i k \cdot x, \tag{4.17}$$

is fortunately very simple. We have already remarked that $g(k)$ contains only singularities in the regions $k^2 < 0$, $k^0 > 0$, and $k^2 > 0$. If these singularities all lie in the lower half of the complex $k^0$ plane, then $I^2(x)$ vanishes for $x^0 < 0$. Similarly, if they lie in the upper half plane, then $I^2(x)$ vanishes for $x^0 > 0$. This follows directly from an evaluation of $I^2$ as a contour integral in the complex $k^0$ plane, a procedure justified by the meromorphic nature of $g(k)$. Thus, in order that $I^2(x_2 - x_1)$ give contributions only for $x_2$ within or on the forward light cone of $x_1$, it is necessary that $g(k)$ should have poles only at points $k^0 = K^0$, where $Im\, K^0 < 0$ if $Re\, K^0 > 0$. If we had arranged conditions so that the meson absorption occurred at $x_1$ and its emission at $x_2$, then the requirement would have been $Im\, K^0 > 0$ if $Re\, K^0 < 0$. However, since the sign of $k^0$ may be changed by a Lorentz transformation only if $|k^0| < |\boldsymbol{k}|$, then our requirements also become sufficient if we demand further that no singularities exist with $Re\, K^0 < |\boldsymbol{k}|$. Since poles a finite distance from the real axis give rise to terms which are damped exponentially, the above restriction should only involve those poles near this axis.

The results of the analysis presented in this section may be summarized as follows. Suppose that the singularities of the propagation function $\Delta'_F(k)$ lie in the second and fourth quadrants of the complex $k^0$ plane, and at least a distance $k$ from the imaginary axis, and that the function $\varrho(k)$ has continuous derivatives of the first $n$ orders. Then the integral $I(x_2 - x_1)$ is composed of two terms, one of which is different from zero only for $(x_2 - x_1)^2 < 0$, $x_2^0 - x_1^0 > 0$, and the other of which decreases in any space-like direction or time-like direction more rapidly than the inverse $n + 4$ power of $|\boldsymbol{x}_1 - \boldsymbol{x}_2|$ or $x_1^0 - x_2^0$, respectively. Furthermore, the decrease of this second term along the surface $|x^0| = r$ is sufficiently rapid so that its integral with respect to $x_2$ over some small region centered at $\langle x_2 \rangle$ decreases as $|\boldsymbol{x}_1 - \langle \boldsymbol{x}_2 \rangle|^{n+3}$.

## 5. Discussion.

The results of the previous section point out rather clearly the distinction between the work of Fierz, and that of Bloch and of Chrétien and Peierls. The basis of the argument of Fierz is that, save for a part which damps out rather rapidly, the positive frequency part of the Feynman Green's function $\Delta_F$ propagates only into the forward light cone. The part which damps out is unobservable due to the complementarity existing between time and energy. This result is essentially dependent upon the location of the poles of the propagation function in the complex $k^0$ plane. By the analysis given here, we find that our integral $I^2$, which is obtained from a process selecting only positive frequency components of the propagation function, also represents a signal propagating only into the forward light cone. The difference between the analysis of Fierz and ours is that the unobservable damped-out term he obtains is, in our case, included in the integral $I^1$.

The analyses of Bloch and Chrétien and Peierls, on the other hand, are mainly concerned with the effect of discontinuities in the form factor. We have found it convenient to include such discontinuities in the integral $I^1$. It might be suspected, then, that their analyses are in some way comparable to that which we gave for $I^1$. This is true in a formal sense if we generalize the interpretation given to the "source function" introduced by Chrétien and Peierls. The physical interpretation given to their integral containing the form function is considerably different from that which we have attached to ours, however. A type of connection between the two may be established, nonetheless. For this purpose we define a four-point "form factor", $F(1346)$, by

$$F(1346 = \int d(25)\; F(123)\; \Delta_F''(2-5)\, F(456), \qquad (5.1)$$

in which $\Delta_F''(2-5)$ is just that part of the propagation function remaining after subtracting off the singularities, in the manner of the last section. Then we may say that our demonstration that $I^1(x_2 - x_1)$ decreases rapidly with increasing distance $|\boldsymbol{x}_2 - \boldsymbol{x}_1|$ or $x_2^0 - x_1^0$ is somewhat equivalent to showing that $F(1346)$ decreases rapidly as the distance from the points 1 and 3 to the points 4 and 6 increases. More exactly, and following the notation of Chrétien and Peierls, we show that, for functions $\varphi(46)$ which are appreciably different from zero only when 4 and 6 are near the origin, $\hat{\varphi}(13)$ decreases as a certain inverse power of the distance of 1 and 3 from the origin, where $\hat{\varphi}(13)$ is defined by

$$\hat{\varphi}(13) = \int d(46)\; F(1346)\, \varphi(46). \qquad (5.2)$$

We found that the power of decrease of $\hat{\varphi}(13)$ depended upon the degree of smoothness of the Fourier transform of $F(1346)$. Written in this manner, the similarity between our investigation of $I^1$ and the investigation by Chrétien and Peierls of the function

$$\tilde{\varphi}(13) = \int d(2)\, F(123)\, \varphi(2) \qquad (5.3)$$

is rather obvious. The different methods used for obtaining conditions on the asymptotic expansions is purely a matter of preference. In view of this similarity, it is not surprising that substantially the same condition is obtained here as was obtained by Chrétien and Peierls.

It seems fairly clear that, for a field theory with non-local interaction, two rather different types of conditions are obtained, both of which must be satisfied for causal behaviour. The first relates to the location of singularities, demanding that they occur only in the second and fourth quadrants in the $k^0$ plane, and at least a distance $|\boldsymbol{k}|$ from the imaginary axis. This type of condition must also be satisfied for a local theory. In practice it restricts the particular choice of a Green's function.

The presence of a non-local interaction, however, introduces an additional amount of freedom into the theory, by means of the form function $G(l_1, l_3)$, which is not completely determined. This in turn creates the possibility for introducing discontinuous factors into the integrands of the integrals giving matrix elements for certain processes. These discontinuities will in general give rise to a type of acausal behaviour, unless the function $G(l_1, l_3)$, considered as a function of the variables $l_1^2, l_3^2, (l_1 + l_3)^2$, is sufficiently smooth. The probability for observing signals transmitted with velocities greater than that of light decreases essentially more rapidly than an inverse $4 + 6$ power of the spatial distance between the points of observation, if the $G$ function has continuous deriviatives of the $n$th order.

The particular problem which we encounter in practice is that we wish, for reasons of convergence, to use form factors which vanish if either $l_1^2, l_3^2$ or $(l_1 + l_3)^2$ is greater than zero. Certainly then $G$ may not be an analytic function of these variables. On the other hand, we may construct a $G$ fulfilling these conditions, and yet possessing continuous derivatives of any preassigned finite order. Thus we may require the "acausal signals" to decrease more rapidly than as any pre-assigned finite inverse power. This is the extent to which causality may be preserved in our theory with non-local interaction.

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## References.


(1) P. Kristensen and C. Møller, Dan. Mat. Fys. Medd. **27**, no. 7 (1952).
(2) C. Bloch, Dan. Mat. Fys. Medd. **27**, no. 8 (1952).
(3) P. Kristensen, private communication.
(4) W. Chrétien and R. E. Peierls, Nuovo Cimento **10**, 668 (1953).
(5) M. Fierz, Helv. Phys. Acta **23**, 731 (1950).
(6) N. van Kampen, Phys. Rev. **89**, 1072; **91**, 1267 (1953).
(7) W. Zimmermann, Zs. f. Phys. **135**, 473 (1953).
(8) C. Källén, Ark. f. Fys. **2**, 371 (1951).
(9) C. N. Yang and D. Feldman, Phys. Rev. **76**, 972 (1950).
(10) H. F. Willis, Phil. Mag. **39**, 455 (1948).



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